Mathematics: Proofs, Models, and Problems, Part 2 (0064)

Test Prep Kit Available
See Inside Back Cover

Test at a Glance

Test Name	**Mathematics: Proofs, Models, and Problems, Part 2**	
Test Code	**0064**	
Time	**1 hour**	
Number of Questions	**3 exercises: 1 advanced proof or model, 1 basic problem, and 1 advanced problem**	
Format	**Constructed-response questions, graphing calculator required**	
	Basic Content Categories	Advanced Content Categories
	Arithmetic and Basic Algebra Geometry Analytic Geometry Functions and Their Graphs Probability and Statistics (without Calculus) Discrete Mathematics	Functions and Their Graphs Probability and Statistics Calculus Discrete Mathematics Abstract Algebra Linear Algebra

About this test

The Mathematics: Proofs, Models, and Problems, Part 2, test is designed to assess the mathematical knowledge and competencies necessary for a beginning teacher of secondary school mathematics. Designed to conform to the curriculum, evaluation, and professional standards of the National Council of Teachers of Mathematics, the test focuses on problem solving, communication, reasoning, and mathematical connections.

This test requires an examinee to construct an advanced model or a proof and solve one basic problem and one advanced problem. The test assesses knowledge in at least three of the content categories listed above. (For a description of basic mathematics content, see Content Knowledge Test 0061.) Competencies from other areas may be required in the course of solving problems.

A question based on content that is listed as both basic and advanced

- will be classified as basic if it requires the examinee to demonstrate an understanding of basic concepts and their applications
- will be classified as advanced if it requires the examinee to demonstrate a deeper conceptual or theoretical understanding.

The basic problem and the advanced problem are each worth 30 percent of the total test score; the advanced model or proof represents 40 percent of the total test score.

The three exercises require the ability to understand and work with mathematical concepts, to reason mathematically, to integrate knowledge of different areas of mathematics, and to develop mathematical models of real-life situations. Constructing a response to each exercise demonstrates the ability to present a solution to a mathematical problem or to explain a mathematical idea in a manner that is correct, complete, clear, and coherent. *Graphing calculators without QWERTY (typewriter) keyboards are required for this test.* For a description of the minimum capabilities required of the calculator, see the Graphing Calculator section of Content Knowledge Test 0061 on page 17.

Selected notations, formulas, and definitions are printed in the test book and are also listed on pages 20-22.

Using the Graphing Calculator

If you use a graphing calculator in answering a question, your response must include a mathematical description of what you use the calculator to do rather than the keystrokes you use. When you use the graphing calculator, keep in mind that the graph produced in the viewing window may not give sufficient information to answer the question due to such factors as screen resolution and the fact that any part of the graph of the function beyond the viewing window is not shown (e.g., the graph of the function $y = x^2 + 10x - 11$ on the viewing window $[-10,10] \times [-10,10]$ looks like a line instead of a parabola and the graph of the function $y = \sin\frac{1}{x}$ on the viewing window $[-10,10] \times [-10,10]$ will not show the true behavior of the function around $x = 0$.

Mathematics Content Descriptions — Advanced

In addition to the Basic Mathematics Content (see Test 0061), the following topics may be covered in the advanced content categories for the Proofs, Models, and Problems, Part 2. Because the assessments were designed to measure the ability to integrate knowledge of mathematics, answering any question may involve more than one competency and may involve competencies from more than one content area.

Functions and Their Graphs

Competencies for this category are listed on page 18.

Probability and Statistics

- Explain the consequences of the Central Limit theorem and why it establishes the importance of the normal distribution in the study of statistics

Calculus

- Prove via an epsilon-delta proof that the limit of a function is actually equal to the calculated value
- Determine the interval of convergence of a power series
- Determine the Taylor series of functions, such as $\sin x$, e^x, and $\ln x$

Discrete Mathematics

- Understand the concept of countability as related to infinite sets
- Find and use finite differences of a function

Abstract Algebra

- Determine if a particular set together with a given operation (or operations) is a group (or ring, or field)
- Use the definition of a group or field to deduce elementary properties of these structures
- Determine whether various mathematical systems (e.g., rational numbers, matrices, motions of an equilateral triangle) are groups, rings, or fields

Linear Algebra

- Determine properties of a set of vectors in a finite dimensional vector space: linear independence, linear span, orthogonality, forming a subspace, forming a basis
- Determine the effects of a linear transformation on a vector space

In addition to the basic competencies in Mathematical Reasoning and Modeling, questions can involve the following additional competencies

- Determine whether two mathematical systems are isomorphic
- Know and use basic facts about non-Euclidean geometries and the ramifications of the three postulates pertaining to the existence of parallel lines

Mathematics: Proofs, Models, and Problems, Part 2 (0064)

This section presents sample questions and constructed-response samples along with the standards used in scoring the advanced mathematical exercises in proofs, models, and problems (for sample questions in basic mathematical exercises, see pages 16-35). When you read these sample responses, keep in mind that they will be less polished than if they had been developed at home, edited, and carefully presented. Examinees do not know what questions will be asked and must decide, on the spot, how to respond. Readers take these circumstances into account when scoring the responses.

Readers will assign scores based on the following scoring guide.

SCORING GUIDE

5

- Clearly demonstrates a full understanding of the mathematical content necessary to answer all parts of the question successfully
- Gives a correct and complete response but may contain a minor calculation error

4

- Clearly demonstrates a full understanding of the mathematical content needed to answer all parts of the question successfully
- *Either* gives a complete response that contains a minor mathematical error or misstatement
  OR gives a correct and almost complete response

3

For a one-part question:

- Clearly demonstrates an understanding of all aspects of the question
- Demonstrates the ability to determine an appropriate strategy for answering the question
- Makes substantial progress toward a correct and complete response

For a multipart question:

- Clearly demonstrates a full understanding of the mathematical content needed to answer a significant portion of the question successfully
- Gives a correct and complete response to that portion of the question.

2

For a one-part question:

- *Either* demonstrates a limited understanding of the question
  OR makes only minimal progress toward a correct and complete response

For a multipart question:

- Clearly demonstrates a full understanding of the mathematical content needed to answer a minor portion of the question successfully
- Gives a correct and complete response to that portion of the question

1

- Demonstrates a very limited understanding of the question
- Makes little or no progress toward a correct and complete response

0

- Blank or off topic

Sample Question 1: Problems

(A) What is the sum of the positive integers 1 through 2,000, inclusive?

(B) What is the sum of the positive integers 1 through 2,000, inclusive, if all multiples of 13 and all multiples of 31 are excluded from the sum?

Sample Response That Received a Score of 5:

(A) $\sum = \frac{n(n+1)}{2} = \frac{(2000)(2001)}{2} = 2,001,000$

(B) $13\overline{)2000}$ = 153.8...

$\sum_{13\text{'s}} = \frac{xn(n+1)}{2} = \frac{13(153)(154)}{2} = 153,153$

$31\overline{)2000}$ = 64.5...

$\sum_{31\text{'s}} = \frac{xn(n+1)}{2} = \frac{31(64)(65)}{2} = 64,480$

13-prime
31-prime > LCM = 403

$403\overline{)2000}$ = 4.9

$\sum_{403\text{'s}} = \frac{xn(n+1)}{2} = \frac{403(4)(5)}{2} = 4,030$

$\sum = \sum_1 - \sum_{13\text{'s}} - \sum_{31\text{'s}} + \sum_{403\text{'s}}$

$= 2,001,000 - 153,153 - 64,480 + 4,030$

$= 1,787,397$

Sample Response That Received a Score of 2:

(A) $\sum_{n=1}^{2000} n$

(B) $\sum_{n=1}^{2000} n - \sum_{n=1}^{153} 13n - \sum_{n=1}^{64} 31n$

$13\overline{)2000}$ = 153: 2000 − 13 → 70; 70 − 65 → 50; 39

$3\overline{)2000}$ = 64: 2000 − 186 → 140; 140 − 124 → 16

Sample Question 2: Problems

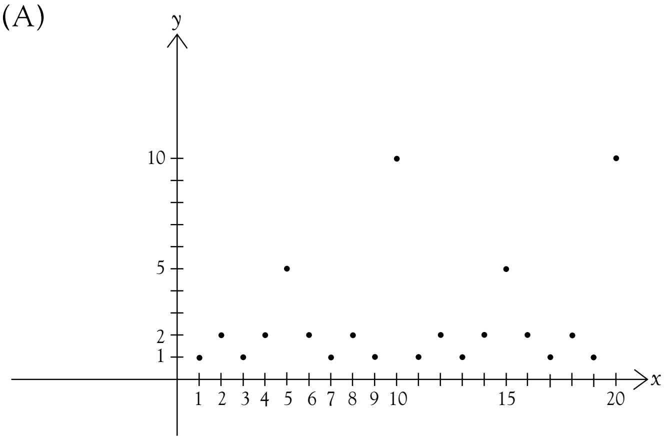
(A) Graph the function $g(x) = gcd\,(10, x)$, where x is a positive integer.

(B) What is the range of $g(x)$?

(C) If a positive integer x is chosen at random, assign a probability that $g(x) = r$ for each r in the range of $g(x)$. Justify your assignment.

Sample Response That Received a Score of 5:

(A)



(B) Range of $g(x) = \{1, 2, 5, 10\}$

(C) $P(g(x) = 1) = \dfrac{4}{10}$

$P(g(x) = 2) = \dfrac{4}{10}$

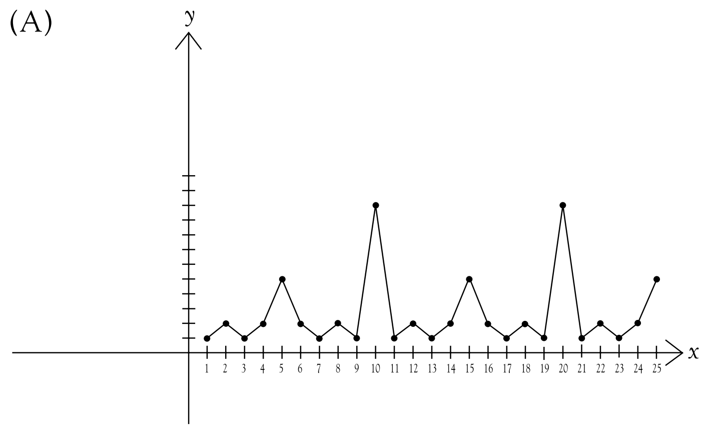
$P(g(x) = 5) = \dfrac{1}{10}$

$P(g(x) = 10) = \dfrac{1}{10}$

The pattern of the elements in the range of $g(x)$ repeats every 10 integral values of x. In each 10-segment section of $g(x)$, we find 1 value of 10, 1 value of 5, 4 values of 2, and 4 values of 1. Thus, 1 in every 10 values is "10"; 1 in every 10 values is "5"; 4 in every 10 values is "2"; and 4 in every 10 values is "1".

Sample Response That Received a Score of 2:

(A)



(B) [1, 10]

Sample Question 3: Problems

Show that if A is a 2×2 invertible matrix all of whose entries are integers and $det(A^{-1})$ is an integer, then all of the entries of A^{-1} are integers.

Sample Response That Received a Score of 5:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $det\,A = ad - cb$, and if a, b, c, d are integers, then $ad - cb$ is an integer.

$det\,A^{-1} = \dfrac{1}{ad - bc}$

If $\dfrac{1}{ad - bc}$ is an integer, then $ab - bc = \pm 1$.

$A^{-1} = \dfrac{1}{det\,A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so $A^{-1} = \begin{pmatrix} \pm d & \pm(-b) \\ \pm(-c) & \pm a \end{pmatrix}$: all entries are integers.

Sample Response That Received a Score of 1:

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

$$\left|\begin{matrix} 3 & 4 \\ -1 & -1 \end{matrix}\right|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & \frac{4}{3} \\ 0 & \frac{1}{3} \end{matrix}\right|\begin{matrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 1 \end{matrix}\right|$$

$$\left|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right|\begin{matrix} -1 & -4 \\ 1 & 3 \end{matrix}\right|$$

$$A^{-1} = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$

-4, -1, 1, and 3 are integers.

Sample Question 4: Problems

Let G be a group with respect to the operation $*$, let $a \in G$, and let $f: G \to G$ be defined by $f(g) = a*g$ for every $g \in G$.

(A) Show that f is one-to-one.
(B) Show that f maps G onto G.

Sample Response That Received a Score of 5:

(A) Assume f is not $1-1$, then there are two elements of G that map to the same element of G under f.

$f(b) = f(c)$
$a * b = a * c$
G is a group therefore a^{-1} exists
$a^{-1} * (a * b) = a^{-1} * (a * c)$
$(a^{-1} * a) * b = (a^{-1} * a) * c$
$b = c$

The two different elements from the assumption are equal, so the assumption was false. Therefore, f is $1-1$.

(B) f is onto if for any $h \in G$, $h = a * c$ for $c \in G$.

$a^{-1} * h = a^{-1} * (a * c)$
$a^{-1} * h = c$
So f is onto.

Sample Response That Received a Score of 1:

(A) Let $a = 2$
$2 \cdot 3 = 6$
$2 \cdot 5 = 10$
$2 \cdot 7 = 14$
} not the same, so multiplication is $1-1$.

(B) $2 \cdot 1 = 2$
$2 \cdot 2 = 4$
$2 \cdot 3 = 6$
$2 \cdot 1\frac{1}{2} = 3$ etc.
$2 \cdot \frac{1}{2} = 1$

Can get all numbers by multiplication.

Sample Question 5: Problems

Invertible 2 × 2 matrices with determinant 1 have the form

$$\begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix} \text{ where } a \neq 0 \text{ and } bc \neq -1$$

or $$\begin{pmatrix} 0 & b \\ \frac{-1}{b} & d \end{pmatrix} \text{ where } b \neq 0 \text{ and } d \neq 0$$

or $$\begin{pmatrix} a & b \\ \frac{-1}{b} & 0 \end{pmatrix} \text{ where } b \neq 0$$

Determine the form(s) of all 2 × 2 matrices that are their own inverses.

Sample Response That Received a Score of 5:

If $A = A^{-1}$, $|A| = \frac{1}{|A|}$ so $|A| = \pm 1$.

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|A| = ad - bc = 1$.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

If $A = A^{-1}$, then $\left.\begin{matrix} b = -b \\ c = -c \end{matrix}\right\}$ $b = c = 0$ and $a = d$.

So $|A| = a^2 = 1$, so $a = \pm 1$.

Then $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(ii) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|A| = ad - bc = -1$.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

If $A = A^{-1}$, then $a = -d$, so
$|A| = -a^2 - bc = -1$
$bc = 1 - a^2$

If $b \neq 0$, $c = \frac{1 - a^2}{b}$

So $A = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$, $b \neq 0$

If $b = 0$, then $|A| = -a^2 = -1$, so $a = \pm 1$; and c is arbitrary.

So $A = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$ or $A = \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}$

So the forms of 2 × 2 matrices that are their own inverses are: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}$,

$\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}$ where $b \neq 0$; a, c any number.

Sample Response That Received a Score of 1:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Sample Question 6: Problems

Let f be a function defined on the real numbers.

Condition 1: $f'(x) = 2x^2 - 5x + 4$
Condition 2: f has exactly 1 real zero, and the zero is between 3 and 4.

(A) Give an example of a function f satisfying condition 1. Show how you arrived at your answer.

(B) Give an example of a function f satisfying both condition 1 and condition 2. Explain how you arrived at your answer.

Sample Response That Received a Score of 5:

(A) $f(x) = \int f'(x)\,dx = \int (2x^2 - 5x + 4)\,dx = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 4x$

(B) By graphing the function in (A) and tracing, I can see that when $x = 3.3$, $f(x)$ is about 10. So the function of $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 4x - 10$ has its root between 3 and 4. I know this is the only root b/c $f'(x) > 0$ for every value of x so the slope of $f(x)$ is always positive.

Sample Response That Received a Score of 2:

(A) $f(x) = \int (2x^2 - 5x + 4)\,dx = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 4x$

(B) The function f in part (A) has a root at $x = 0$, so the function $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 + 4x + 3.5$ will have a root at $x = 3.5$.

Sample Question 7: Proof

(A) A sequence is defined by $a_1 = 2$ and $a_{n+1} = 3 + a_n$, for n 1. Express a_n in closed form.

(B) Prove by mathematical induction that the closed form of a_n in your answer to (A) is correct.

(C) Explain how the principle of mathematical induction is used to prove theorems.

Sample Response That Received a Score of 5:

(A) $a_1 = 2$
$a_2 = 3 + 2$
$a_3 = 3 + (3 + 2) = 2 \bullet 3 + 2$
$a_4 = 3 + (3 + (3 + 2)) = 3 \bullet 3 + 2$
.
.
.
$a_n = (n - 1) \bullet 3 + 2$

(B) OK if $n = 2$: $a_2 = 1 \bullet 3 + 2$. Assume OK for $a_n = (n - 1) \bullet 3 + 2$. Prove OK for a_{n+1}:

$a_{n+1} = 3 + a_n = 3 + (n - 1) \bullet 3 + 2$
$= n \bullet 3 + 2$, so OK for all n.

(C) Mathematical induction needs 2 steps:

Step 1: Show that the theorem is true for a certain value of n, say $n = 1$.

Step 2: Assume that the theorem is true for $n = k$. Then show that it must also be true for $n = k + 1$. Then the theorem is true for all n, because it is true for $n = 1$ (from step 1), it is true for $n = 1 + 1 = 2$ (from step 2), it is true for $n = 2 + 1$, etc.

Sample Response That Received a Score of 1:

(A) $a_1 = 2$
$a_2 = 3 + 2 = 5$
$a_3 = 3 + 5 = 8$
$a_4 = 3 + 8 = 11$
$a_5 = 3 + 11 = 14$

Sample Question 8: Proof

Show that if F is a field with more than one element then the additive identity cannot have a multiplicative inverse.

Note: The definition of a field will be included in the examination booklet.

Sample Response That Received a Score of 5:

Suppose the additive identity is 0 and the multiplicative identity is 1. Suppose 0 has a multiplicative inverse 0^{-1}.

Then $0^{-1} \bullet 0 = 1$.

But $0 + 0 = 0$

and $0^{-1} \bullet (0 + 0) = 0^{-1} \bullet 0$

$0^{-1} \bullet 0 + 0^{-1} \bullet 0 = 0^{-1} \bullet 0$

$\quad 1 \quad + \quad 1 \quad = \quad 1$

not true because 1 is not the additive identity. So the assumption that 0 has a multiplicative inverse was false.

Sample Response That Received a Score of 1:

F is a field that contains $\{a_1, a_2, a_3, a_4, \ldots a_n\}$

Additive identity is 0

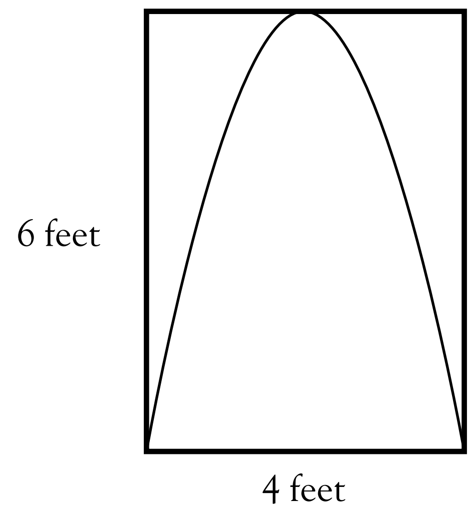
$0 \times 0 = 1$

$0 \times 5 = 1$ are impossible equations.

$0 \times 8 = 1$

Therefore, 0 cannot have a multiplicative inverse.

Sample Question 9: Modeling

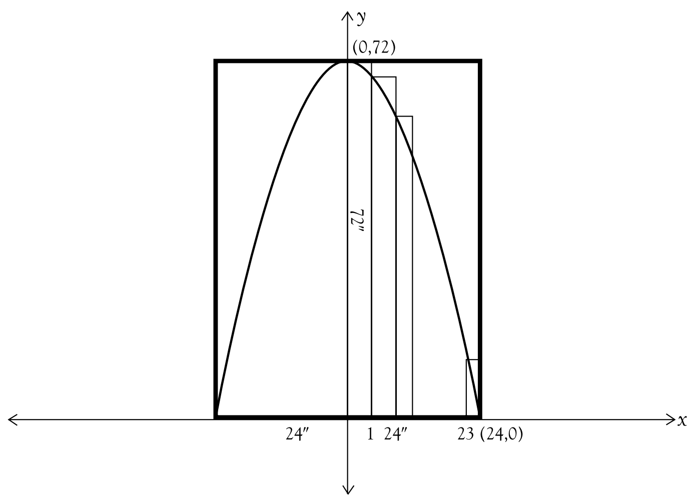


As part of a modern painting, an artist is going to cover completely the parabolic region shown in the rectangle above with nonoverlapping 1-inch wide strips of gold-colored paper.

(A) Approximately how many feet of the gold-colored strips will be needed?

(B) Explain why the process by which you solved the problem in (A) is an appropriate process to use.

Sample Response That Received a Score of 5:



(A) & (B) Draw $x - y$ axis as shown. The parabola has the equation

$$y = a(x - b)^2 + p$$
$$y = a(x - 0)^2 + 72$$
$$y = ax^2 + 72 \qquad \text{find } a\text{: since } (24, 0) \text{ is on the graph}$$
$$0 = a(24)^2 + 72$$
$$\frac{-72}{576} = a \qquad a = \frac{-1}{8} \qquad y = -\frac{x^2}{8} + 72$$

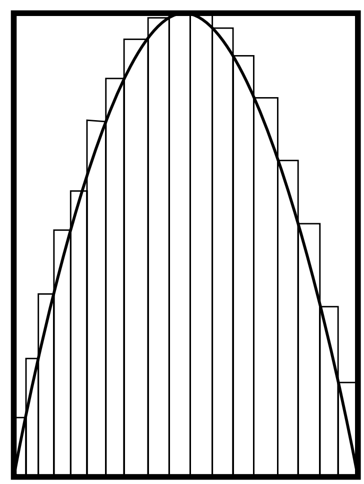
S = Sum of the heights of the rectangles

$$S = 2\left[\frac{-1}{8}(0)^2 + 72 + \frac{-1}{8}(1)^2 + 72 + \frac{-1}{8}(2)^2 + 72 + \frac{-1}{8}(3)^2 + 72 + \ldots \frac{-1}{8}(23)^2 + 72\right]$$
$$S = 2\left[(72)(24) + \frac{-1}{8}(1^2 + 2^2 + 3^2 + 4^2 + \ldots 23^2)\right]$$
$$S = 2\left[1728 - \frac{1}{8}\,\frac{(23)(24)(47)}{6}\right]$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$S = 3456 - 1081$$

The artist needs 2375 inches of 1 inch wide strips.

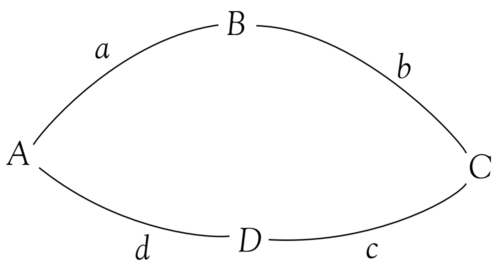
$$S = 2375$$

Sample Response That Received a Score of 2:



Divide the 48″ base into 48 pieces each of width 1″. Draw the rectangles (strips) as shown in the figure above. Since half the strips are longer than $\frac{1}{2}(72'') = 36''$ and half the pieces are shorter than 36″, we can use 36″ for the length of each strip. 36″(48) = 1728 inches of colored paper strips are required.

Sample Question 10: Modeling



The figure above represents the only roadways connecting four towns *A*, *B*, *C*, and *D*.

After a severe rainstorm the probability that each of the roadways will be passable is estimated to be as follows:

Roadways	Probability Roadway Passable
a	0.35
b	0.40
c	0.45
d	0.42

The knowledge that one roadway is passable in no way helps to predict whether any other roadway will be passable.

(A) What is the probability that, after a severe rainstorm, a driver will be able to drive from Town A to Town C using roadways *a* and *b*? Explain your reasoning in calculating this probability.

(B) What is the probability that, after a severe rainstorm, a driver will be able to drive from Town A to Town C? Explain your reasoning in calculating this probability.

Sample Response That Received a Score of 5:

(A) Driver can get from A to C using *a* and *b* if both are open. The probabilities are independent so $P = (.35)(.4) = .14$

(B) If *a* and *b* are open then the driver can get from A to C no matter what the condition of the other two roads. Similarly for *c* and *d*.

$P(a \text{ and } b \text{ open}) = .14$
$P(c \text{ and } d \text{ open}) = .45(.42) = .189$

Sum is .329

But in both probabilities all 4 roadways may be open. Counted twice. Must be subtracted out of .329.

$P(\text{all open}) = (.35)(.40)(.45)(.42) = .02646$

P can get from A to C $= .329 - .02646 = .30254$

Sample Response That Received a Score of 2:

(A) $P(a \text{ is open}) = .35$ and $P(b \text{ is open}) = .4$
Since the probabilities are independent, the probability that both are open is .14, because that equals (.35)(.4).

(B) *{Blank}*